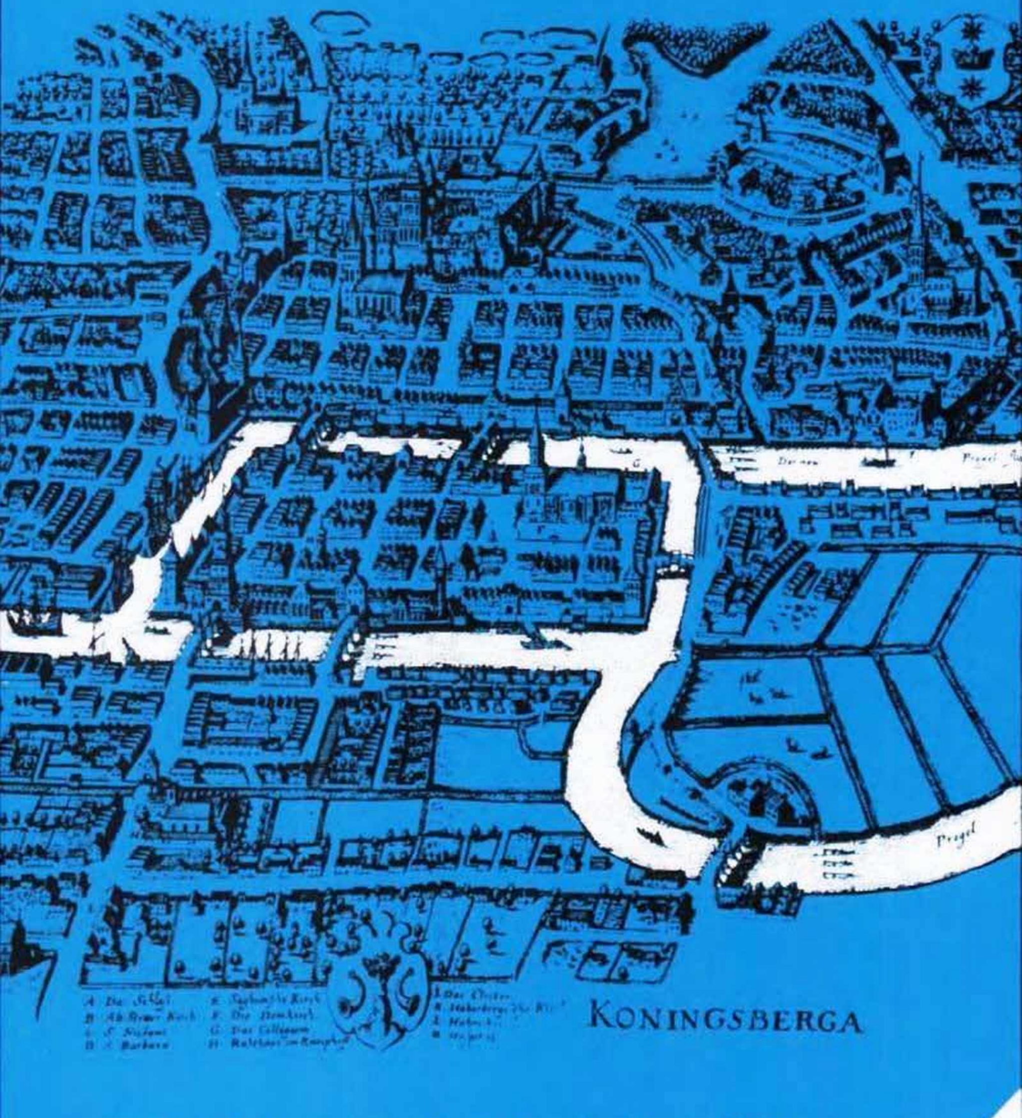
GRAPH THEORY
1736-1936



N. L. Biggs, E. K. Lloyd, R. J. Wilson

REISSUE

Graph Theory

1736–1936

Hamilton's Icosian Game (Chap. 2)

Graph Theory

1736-1936

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L. EULER (1707–83)

D. KÖNIG (1884–1944)

Preface

MATHEMATICIANS have often pursued their researches in an erratic and intuitive way, rather than by the clear light of logic; consequently, the historical development of the subject frequently differs considerably from the systematic approach which one finds in most textbooks. In this book we shall follow an historical approach, and give a self-contained introduction to the subject of graph theory. We hope that the reader will thereby come to appreciate the complex web of ideas and influences which come together to form a mathematical theory.

Our decision to cover the period 1736–1936 is the result of a convenient historical accident. In 1736 the first article on a topic relating to graph theory was written by the Swiss mathematician Leonhard Euler; just two hundred years later, in 1936, the first full-length book on the subject, written by Dénes König, was published. Of course, graph theory did not stop in 1936, and we have not felt obliged to exclude all reference to later work in the subject.

The central feature of this book is a set of thirty-seven extracts taken from the original writings of mathematicians who contributed to the foundations of graph theory. Where necessary, these extracts have been translated into English, and they have been edited by the omission of certain sections, but no other significant changes have been made. A list of these extracts may be found on pages (ix)–(x).

The book has ten chapters. Each chapter deals with a particular theme in graph theory, and contains three or four main extracts; these extracts are linked by a commentary which traces the historical development of the theme. Superimposed on this structure is the framework of a conventional textbook, wherein the relevant mathematical terminology and notation are explained, in logical progression, as they are required.

There are a few conventions regarding the organization of this book which may need some explanation. The references within each chapter (including those references which occur in the extracts) are labelled in a single numerical sequence, and they are listed at the end of the chapter. The only exceptions to this rule are the extracts themselves, which are referred to by their chapter and letter. The figures in each chapter are similarly labelled in a single sequence. Terminology relating to graph theory, defined in the commentary, is printed in bold type; definitions which occur in the extracts, and those which occur in the commentary but which are not directly relevant to graph theory, are printed in italics. The omission of a section from an abstract is signified by a row of five asterisks; there are also occasional omissions of words or phrases, and these are indicated by five dots. Additions or alterations to the extracts are enclosed in square brackets.

At the end of the book there are three appendices. The first of these gives a brief account of the development of graph theory since 1936. The second appendix contains biographical information about the main characters in our story; the reader should note that capital letters are used in the text for the first mention of all those whose biographies are given in this appendix. The third appendix is a comprehensive bibliography of the work on graph theory published in the period 1736–1936.

A book of this kind could not be written without help, in the form of advice, criticism, and information, from many friends and colleagues. Our special thanks are due to P. J. Federico, who has spent several years working on the history of graph theory. He is preparing a book on the subject which, however, will differ from ours, both in the period covered and in the treatment of the material. Mr. Federico has been generous enough to provide us with drafts of his work, and he has commented in detail on our text. We have pleasure in thanking him for his assistance.

Our thanks are also due to many others who have assisted our work, and especially to D. D. V. Morgan, G. de Barra, R. V. Turley, M. Askew and the staffs of the various libraries who have unearthed a number of obscure journals to meet our requests. We also thank the secretarial staffs of the Mathematics Departments of Royal Holloway College and the Open University for their help with typing the various drafts of the manuscript.

* * *

In this reprint (1998) the text has been modified to take account of recent historical scholarship, and the reference sections have been expanded and updated.

Contents

List of extracts

List of plates

(The plates will be found between pages 208 and 209.)

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1

Paths

J. B. LISTING (1808–82)

THE origins of graph theory are humble, even frivolous. Whereas many branches of mathematics were motivated by fundamental problems of calculation, motion, and measurement, the problems which led to the development of graph theory were often little more than puzzles, designed to test the ingenuity rather than to stimulate the imagination. But despite the apparent triviality of such puzzles, they captured the interest of mathematicians, with the result that graph theory has become a subject rich in theoretical results of a surprising variety and depth.

In this chapter we shall be concerned mainly with the origin and ramifications of one particular puzzle—the problem of the Königsberg bridges. The solution of this problem involves the formulation of several of the basic concepts of graph theory.

The problem of the Königsberg bridges

The map in Fig. 1.1 is taken from a book [1] published in the seventeenth century. It is an artist's impression of the old city of Königsberg in Eastern Prussia, showing the River Pregel which flows through the city. As can be seen, the Pregel surrounds an island (called Kneiphof), and, on the right of the map, it separates into two branches. To enable the citizens of Königsberg to travel easily from one part of the city to another, the river was spanned by seven bridges, with such delightful names as Honey Bridge and Blacksmith's Bridge.

It is said that the people of Königsberg used to entertain themselves by trying to devise a route around the city which would cross each of the seven bridges just once. Since their attempts had always failed, many of them believed that the task was impossible, but it was not until the 1730s that the problem was treated from a mathematical point of view and the

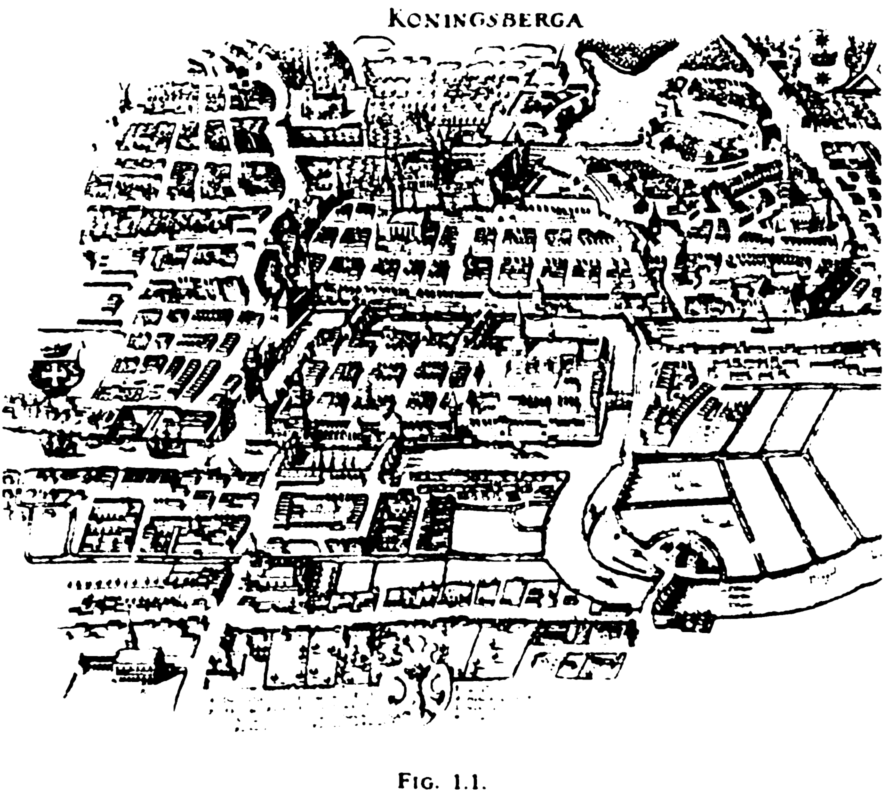

FIG. 1.1.

impossibility of finding such a route was proved. In 1736, one of the leading mathematicians of the time, Leonhard EULER, communicated with other mathematicians on the problem [16], and wrote an article in which he dealt with this particular problem and gave a general method for other problems of the same type. His article was of considerable importance, both for graph theory and for the development of mathematics as a whole, and we shall give a translation of it in full [1A]. However, the reader may prefer to stop after Paragraph 9, and go on to our commentary at the end of the article, since Euler's main results will be proved more succinctly later in the chapter.

1A

L. EULER

SOLUTIO PROBLEMATIS AD GEOMETRIAM SITUS PERTINENTIS

[The solution of a problem relating to the geometry of position]

Commentarii Academiae Scientiarum Imperialis Petropolitanae **8** (1736), 128–140.
(Based on a talk presented to the Academy on 26 August 1735.)

1. In addition to that branch of geometry which is concerned with magnitudes, and which has always received the greatest attention, there is another branch, previously almost unknown, which Leibniz first mentioned, calling it the *geometry of position*. This branch is concerned only with the determination of position and its properties; it does not involve measurements, nor calculations made with them. It has not yet been satisfactorily determined what kind of problems are relevant to this geometry of position, or what methods should be used in solving them. Hence, when a problem was recently mentioned, which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it was concerned with the geometry of position—especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this kind of problem, as an example of the geometry of position.

2. The problem, which I am told is widely known, is as follows: in Königsberg in Prussia, there is an island *A*, called *the Kneiphof;* the river which surrounds it is divided into two branches, as can be seen in Fig. [1.2], and these branches are crossed by seven bridges, *a*, *b*, *c*, *d*, *e*, *f* and *g*. Concerning these bridges, it was asked whether anyone could arrange a route in such a way that he would cross each bridge once and only once. I was told that some people asserted that this was impossible, while others were in doubt; but nobody would actually assert that it could be done. From this, I have formulated the general problem: whatever be the arrangement and division of the river into branches, and however many bridges there be, can one find out whether or not it is possible to cross each bridge exactly once?

3. As far as the problem of the seven bridges of Königsberg is concerned, it can be solved by making an exhaustive list of all possible routes, and then finding

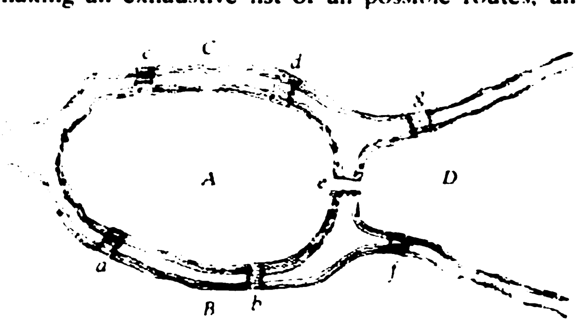

FIG. 1.2.

whether or not any route satisfies the conditions of the problem. Because of the number of possibilities, this method of solution would be too difficult and laborious, and in other problems with more bridges it would be impossible. Moreover, if this method is followed to its conclusion, many irrelevant routes will be found, which is the reason for the difficulty of this method. Hence I rejected it, and looked for another method concerned only with the problem of whether or not the specified route could be found; I considered that such a method would be much simpler.

4. My whole method relies on the particularly convenient way in which the crossing of a bridge can be represented. For this I use the capital letters *A*, *B*, *C*, *D*, for each of the land areas separated by the river. If a traveller goes from *A* to *B* over bridge *a* or *b*, I write this as *AB*—where the first letter refers to the area the traveller is leaving, and the second refers to the area he arrives at after crossing the bridge. Thus if the traveller leaves *B* and crosses into *D* over bridge *f*, this crossing is represented by *BD*, and the two crossings *AB* and *BD* combined I shall denote by the three letters *ABD*, where the middle letter *B* refers both to the area which is entered in the first crossing and to the one which is left in the second crossing.

5. Similarly, if the traveller goes on from *D* to *C* over the bridge *g*, I shall represent these three successive crossings by the four letters *ABDC*, which should be taken to mean that the traveller, starting in *A*, crosses to *B*, goes on to *D*, and finally arrives in *C*. Since each land area is separated from every other by a branch of the river, the traveller must have crossed three bridges. Similarly, the successive crossing of four bridges would be represented by five letters, and in general, however many bridges the traveller crosses, his journey is denoted by a number of letters one greater than the number of bridges. Thus the crossing of seven bridges requires eight letters to represent it.

6. In this method of representation, I take no account of the bridges by which the crossing is made, but if the crossing from one area to another can be made by several bridges, then any bridge can be used, so long as the required area is reached. It follows that if a journey across the seven bridges [of Fig. 1.2] can be arranged in such a way that each bridge is crossed once, but none twice, then the route can be represented by eight letters which are arranged so that the letters *A* and *B* are next to each other twice, since there are two bridges, *a* and *b*, connecting the areas *A* and *B*; similarly, *A* and *C* must be adjacent twice in the series of eight letters, and the pairs *A* and *D*, *B* and *D*, and *C* and *D* must occur together once each.

7. The problem is therefore reduced to finding a sequence of eight letters, formed from the four letters *A*, *B*, *C* and *D*, in which the various pairs of letters occur the required number of times. Before I turn to the problem of finding such a sequence, it would be useful to find out whether or not it is even possible to arrange the letters in this way, for if it were possible to show that there is no such arrangement, then any work directed towards finding it would be wasted. I have therefore tried to find a rule which will be useful in this case, and in others, for determining whether or not such an arrangement can exist.

8. In order to try to find such a rule, I consider a single area *A*, into which there lead any number of bridges *a*, *b*, *c*, *d*, etc. (Fig. [1.3]). Let us take first the single bridge *a* which leads into *A*: if a traveller crosses this bridge, he must either have

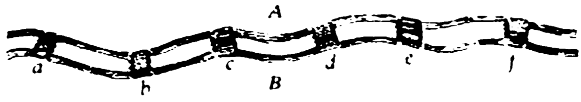

FIG. 1.3.

been in A before crossing, or have come into A after crossing, so that in either case the letter A will occur once in the representation described above. If three bridges (a, b and c, say) lead to A, and if the traveller crosses all three, then in the representation of his journey the letter A will occur twice, whether he starts his journey from A or not. Similarly, if five bridges lead to A, the representation of a journey across all of them would have three occurrences of the letter A. And in general, if the number of bridges is any odd number, and if it is increased by one, then the number of occurrences of A is half of the result.

9. In the case of the Königsberg bridges, therefore, there must be three occurrences of the letter A in the representation of the route, since five bridges (a, b, c, d, e) lead to the area A. Next, since three bridges lead to B, the letter B must occur twice; similarly, D must occur twice, and C also. So in a series of eight letters, representing the crossing of seven bridges, the letter A must occur three times, and the letters B, C and D twice each—but this cannot happen in a sequence of eight letters. It follows that such a journey cannot be undertaken across the seven bridges of Königsberg.

10. It is similarly possible to tell whether a journey can be made crossing each bridge once, for any arrangement of bridges, whenever the number of bridges leading to each area is odd. For if the sum of the number of times each letter must occur is one more than the number of bridges, then the journey can be made; if, however, as happened in our example, the number of occurrences is greater than one more than the number of bridges, then such a journey can never be accomplished. The rule which I gave for finding the number of occurrences of the letter A from the number of bridges leading to the area A holds equally whether all of the bridges come from another area B, as shown in Fig. [1.3], or whether they come from different areas, since I was considering the area A alone, and trying to find out how many times the letter A must occur.

11. If, however, the number of bridges leading to A is even, then in describing the journey one must consider whether or not the traveller starts his journey from A; for if two bridges lead to A, and the traveller starts from A, then the letter A must occur twice, once to represent his leaving A by one bridge, and once to represent his returning to A by the other. If, however, the traveller starts his journey from another area, then the letter A will only occur once; for this one occurrence will represent both his arrival in A and his departure from there, according to my method of representation.

12. If there are four bridges leading to A, and if the traveller starts from A, then in the representation of the whole journey, the letter A must occur three times if he is to cross each bridge once; if he begins his walk in another area, then the letter A will occur twice. If there are six bridges leading to A, then the letter A will occur four times if the journey starts from A, and if the traveller does not start by leaving A, then it must occur three times. So, in general, if the number of bridges is even, then the number of occurrences of A will be half of this number if

the journey is not started from A, and the number of occurrences will be one greater than half the number of bridges if the journey does start at A.

13. Since one can start from only one area in any journey, I shall define, corresponding to the number of bridges leading to each area, the number of occurrences of the letter denoting that area to be half the number of bridges plus one, if the number of bridges is odd, and if the number of bridges is even, to be half of it. Then, if the total of all the occurrences is equal to the number of bridges plus one, the required journey will be possible, and will have to start from an area with an odd number of bridges leading to it. If, however, the total number of letters is one less than the number of bridges plus one, then the journey is possible starting from an area with an even number of bridges leading to it, since the number of letters will therefore be increased by one.

14. So, whatever arrangement of water and bridges is given, the following method will determine whether or not it is possible to cross each of the bridges:

I first denote by the letters A, B, C, etc. the various areas which are separated from one another by the water. I then take the total number of bridges, add one, and write the result above the working which follows. Thirdly, I write the letters A, B, C, etc. in a column, and write next to each one the number of bridges leading to it. Fourthly, I indicate with an asterisk those letters which have an even number next to them. Fifthly, next to each even one I write half the number, and next to each odd one I write half the number increased by one. Sixthly, I add together these last numbers, and if this sum is one less than, or equal to, the number written above, which is the number of bridges plus one, I conclude that the required journey is possible. It must be remembered that if the sum is one less than the number written above, then the journey must begin from one of the areas marked with an asterisk, and it must begin from an unmarked one if the sum is equal.

Thus in the Königsberg problem, I set out the working as follows:

Number of bridges 7, which gives 8

	Bridges	
A,	5	3
B,	3	2
C,	3	2
D,	3	2

Since this gives more than 8, such a journey can never be made.

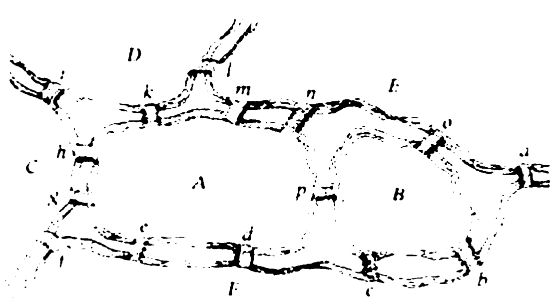

Fig. 1.4.

15. Suppose that there are two islands A and B surrounded by water which leads to four rivers as shown in Fig. [1.4]. Fifteen bridges (a, b, c, d, etc.) cross the rivers and the water surrounding the islands, and it is required to determine whether one can arrange a journey which crosses each bridge exactly once. First, therefore, I name all the areas separated by water as A, B, C, D, E, F, so that there are six of them. Next, I increase the number of bridges (15) by one, and write the result (16) above the working which follows.

		16
A^*,	8	4
B^*,	4	2
C^*,	4	2
D,	3	2
E,	5	3
F^*,	6	3
		16

Thirdly, I write the letters A, B, C, etc. in a column, and write next to each one the number of bridges which lead to the corresponding area, so that eight bridges lead to A, four to B, and so on. Fourthly, I indicate with an asterisk those letters which have an even number next to them. Fifthly, I write in the third column half the even numbers in the second column, and then I add one to the odd numbers and write down half the result in each case. Sixthly, I add up all the numbers in the third column in turn, and I get the sum 16; since this is equal to the number (16) written above, it follows that the required journey can be made if it starts from area D or E, since these are not marked with an asterisk. The journey can be done as follows:

$$EaFbBcFdAeFfCgAhCiDkAmEnApBoElD.$$

where I have written the bridges which are crossed between the corresponding capital letters.

16. In this way it will be easy, even in the most complicated cases, to determine whether or not a journey can be made crossing each bridge once and once only. I shall, however, describe a much simpler method for determining this which is not difficult to derive from the present method, after I have first made a few preliminary observations. First, I observe that the numbers of bridges written next to the letters A, B, C, etc. together add up to twice the total number of bridges. The reason for this is that, in the calculation where every bridge leading to a given area is counted, each bridge is counted twice, once for each of the two areas which it joins.

17. It follows that the total of the numbers of bridges leading to each area must be an even number, since half of it is equal to the number of bridges. This is impossible if only one of these numbers is odd, or if three are odd, or five, and so on. Hence if some of the numbers of bridges attached to the letters A, B, C, etc. are odd, then there must be an even number of these. Thus, in the Königsberg problem, there were odd numbers attached to the letters A, B, C and D, as can be seen from Paragraph 14, and in the last example (in Paragraph 15), only two numbers were odd, namely those attached to D and E.

18. Since the total of the numbers attached to the letters *A*, *B*, *C*, etc. is equal to twice the number of bridges, it is clear that if this sum is increased by 2 and then divided by 2, then it will give the number which is written above the working. If, therefore, all of the numbers attached to the letters *A*, *B*, *C*, *D*, etc. are even, and half of each of them is taken to obtain the numbers in the third column, then the sum of these numbers will be one less than the number written above. Whatever area marks the beginning of the journey, it will have an even number of bridges leading to it, as required. This will happen in the Königsberg problem if the traveller crosses each bridge twice, since each bridge can be treated as if it were split in two, and the number of bridges leading into each area will therefore be even.

19. Furthermore, if only two of the numbers attached to the letters *A*, *B*, *C*, etc. are odd, and the rest are even, then the journey specified will always be possible if the journey starts from an area with an odd number of bridges leading to it. For, if the even numbers are halved, and the odd ones are increased by one, as required, the sum of their halves will be one greater than the number of bridges, and hence equal to the number written above.

It can further be seen from this that if four, or six, or eight ... odd numbers appear in the second column, then the sum of the numbers in the third column will be greater by one, two, three ... than the number written above, and the journey will be impossible.

20. So whatever arrangement may be proposed, one can easily determine whether or not a journey can be made, crossing each bridge once, by the following rules:

If there are more than two areas to which an odd number of bridges lead, then such a journey is impossible.

If, however, the number of bridges is odd for exactly two areas, then the journey is possible if it starts in either of these areas.

If, finally, there are no areas to which an odd number of bridges leads, then the required journey can be accomplished starting from any area.

With these rules, the given problem can always be solved.

21. When it has been determined that such a journey can be made, one still has to find how it should be arranged. For this I use the following rule: let those pairs of bridges which lead from one area to another be mentally removed, thereby considerably reducing the number of bridges; it is then an easy task to construct the required route across the remaining bridges, and the bridges which have been removed will not significantly alter the route found, as will become clear after a little thought. I do not therefore think it worthwhile to give any further details concerning the finding of the routes.

Euler's treatment of the Königsberg problem involved two major steps. First, in Fig. 1.2, he replaced the map of the city by a simple diagram showing its main features, and then, in Paragraphs 4 and 7 of his article, he formulated the problem in such a way that the diagram became

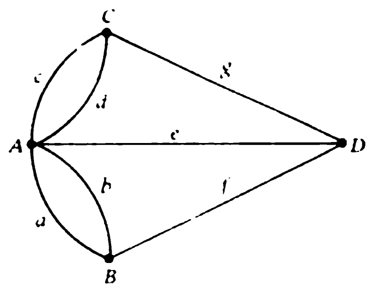

FIG. 1.5.

unnecessary. He denoted the four land areas by the symbols A, B, C, D, and the seven bridges by a, b, c, d, e, f, g, where the bridge a joins A and B, e joins A and D, and so on. This is an example of what we now refer to as a 'graph', and Euler's problem of finding a sequence of eight symbols with a particular property (described in Paragraph 7) is related to the existence of a special kind of 'path' in the graph.

To explain the exact meaning of these terms we must give some definitions. A **graph** consists of a finite set of **vertices**, a finite set of **edges**, and a rule which tells us which edges **join** which pairs of vertices. Normally, an edge joins two distinct vertices, but exceptionally the two vertices may coincide; in the latter case the edge is said to be a **loop**. In our particular example there are four vertices, corresponding to the land areas A, B, C, D, and seven edges, corresponding to the seven bridges; the rule tells us that the edges a and b join the vertices A and B, the edges c and d join the vertices A and C, and so on. We also define a **path** in a graph to be a sequence of vertices and edges,

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{r-1}, e_r, v_r$$

in which each edge e_i joins the vertices v_{i-1} and v_i $(1 \leq i \leq r)$.

It is helpful to illustrate these abstract definitions by representing a graph pictorially. We depict a graph as a diagram of points and lines, in which the points represent vertices and the lines represent edges; a diagram for the Königsberg graph is shown in Fig. 1.5. It should be noted that this is merely a convenient way of describing the graph—we repeat that the graph itself is an abstract entity consisting of the four vertices A, B, C, D, the seven edges a, b, c, d, e, f, g, and the rule which tells us how the edges join the vertices. Nevertheless, the pictorial representation of graphs is a very useful technique and we shall use it throughout this book.

We may now formulate the problem of the Königsberg bridges using the terminology just introduced: the object of the problem is to find a path which contains each edge of the graph once and only once. A path of this kind is now called an **Eulerian path**, and Euler showed that the

Königsberg graph has no such path. He also investigated the existence of Eulerian paths in general graphs.

In order that a graph should contain an Eulerian path, it is clearly necessary that the graph should be **connected**; this means that for any two vertices v and w it is possible to find a path beginning at v and ending at w, so that the graph is 'all in one piece'. Euler took this condition for granted, since it was automatically satisfied in the examples he considered. A **disconnected** graph, that is, one which is not connected, splits up into connected parts, called its **components**.

We need only one more definition at this stage. The **valency** (or **degree**) of a vertex v is the number of edges which meet at v; for example, in the Königsberg graph, the valency of the vertex A is five, and the valencies of B, C, and D are all three. (The reason for the name 'valency' will be given in Chapter 4.) Notice that if we add the valencies of all the vertices in a graph, then the sum is just twice the number of edges in the graph, since each edge contributes twice to the sum. This fact was mentioned in Paragraph 16 of Euler's article, and it yields the useful result (Paragraph 17) that, *in any graph, the number of vertices with odd valency must be even*. This is sometimes referred to as the 'handshaking lemma', since it tells us that if the guests at a party shake hands when they meet, then the number of guests who have shaken hands an odd number of times must necessarily be even.

We now have all the terminology we need to state Euler's main result, given in lines 4 and 5 of Paragraph 20: *if a connected graph has more than two vertices of odd valency, then it cannot contain an Eulerian path*. In the same paragraph, Euler also stated the converse result: *if a connected graph has no vertices of odd valency, or two such vertices, then it contains an Eulerian path*. Unfortunately, Euler did not give a proof of the latter result, presumably because he considered it to be self-evident. This lack of precision was quite common among eighteenth-century mathematicians, and occasionally it led them into the realm of fantasy, as when Euler asserted the truth of the equation

$$1-1+1-1+1-\ldots=\tfrac{1}{2}.$$

Nevertheless, Euler's graph-theoretical intuition was correct, although a complete proof of the converse result did not appear in print until 1873 [1B]. The proof was due to a young German mathematician, Carl Hierholzer, whose work was prepared for publication by a colleague, C. Wiener. The tragic circumstances were explained in a footnote:

> Privatdocent Dr. Hierholzer, unfortunately prematurely snatched away by death from the service of scholarship (died 13 September 1871), reported on the following investigation to a circle of mathematical friends. It was in

order to save it from oblivion (and it had to be reconstructed without any written record) that I sought to complete the following as accurately as possible, with the help of my esteemed colleague Lüroth.

1B

C. HIERHOLZER

ÜBER DIE MÖGLICHKEIT, EINEN LINIENZUG OHNE WIEDERHOLUNG UND OHNE UNTERBRECHNUNG ZU UMFAHREN

[On the possibility of traversing a line-system without repetition or discontinuity]

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In an arbitrary system of interwoven lines, we can define the *branches* at a point to be the distinct lines of the network along which it is possible to leave the point in question. A point at which there are several branches is called a *node*, and is termed as many-fold as the number of branches there, being called odd or even according as this number is odd or even. Thus, an ordinary double point may be called a four-fold node, an ordinary point is a two-fold node, and a free end may be termed a one-fold node.

If a line-system can be traversed in one path without any section of line being traversed more than once, then the number of odd nodes is either zero or two. If, in carrying out this process, we pass through any node, then two of the branches at that node are used, and since no line-segment may be traversed twice, a node which we pass through n times must be a $2n$-fold node. A point can therefore be an odd node only if on one occasion we do not pass through it, that is, if it is an initial or terminal point. If, on reaching the end of the journey, we return to the starting point, then there can only be even nodes; if not, then the initial and terminal points are odd nodes.

Conversely: *if a connected line-system has either no odd nodes or two odd nodes, then the system can be traversed in one path.*

For (a) if only part of the line-system has been traversed, then every node in the remaining part remains even or odd, just as it was in the original system; only the initial and terminal nodes of the traversed part change their parity, unless they coincide. This is because two branches are used in passing through a node, and only one branch is used at the start and finish of the path.

(b) If we begin to traverse the system at an odd node, then we can finish only at another odd node. This is because two branches are used each time we pass through an even node, so that each time we arrive at such a node, there is at least one other branch available to depart along. However, the initial node is converted at the beginning to an even node, so that it is also impossible to stop there. On the other hand, if we start to traverse the system at an even node, then we can also terminate at the same node, since it is changed at the outset to an odd one.

(c) If, now, the system has two odd nodes, then a path beginning at one of them necessarily terminates at the other. In this case the completed part of the path is open. If, on the other hand, the given system has no odd nodes, then a path beginning at any node (which must be an even node) must necessarily terminate at that same node. In this case the completed path is closed.